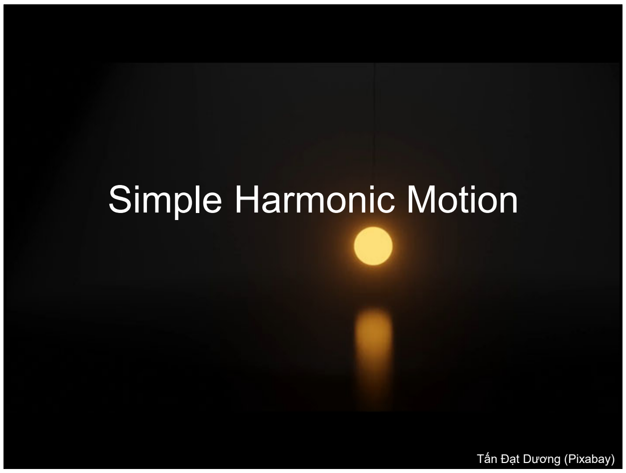
Simple Harmonic Motion



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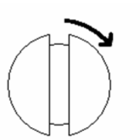
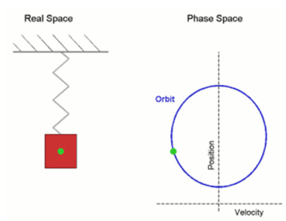
Oscillations

- Repetitive (periodic) motion
  - Object moves around an equilibrium position.





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Period and Frequency

- The time to complete one oscillation remains constant and is called the **period, T**.
- **Frequency, f**, is defined to be the number of oscillations per unit time.
  - Measured in hertz (Hz).
- The relationship between frequency and period is

$$T = \frac{1}{f}$$

Simple Harmonic Motion

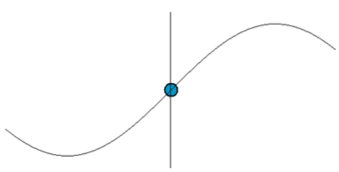
- Simple Harmonic Motion (SHM) is a type of oscillatory motion where the net force can be described by Hooke's law.
  - The accelerating force acts to restore the object to its equilibrium position.

$$F \propto -x$$

  - The acceleration is proportional to the displacement of the object from its equilibrium position.

$$a \propto -x$$

- A simple harmonic oscillator will oscillate with equal displacement on either side of the equilibrium position.
- All simple harmonic motion is intimately related to sine and cosine waves.



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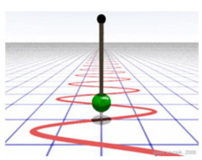

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Displacement

- The displacement as a function of time t in any simple harmonic motion, is given by

$$x = A\cos(2\pi f t)$$

Where: A is the amplitude
f is the frequency of the oscillations

Velocity

- The velocity of the oscillations at a point in time is

$$v = -v_{max}\sin(2\pi f t)$$

Where: $v_{max} = 2\pi f A$

Acceleration

- The acceleration of the oscillations at a point in time is

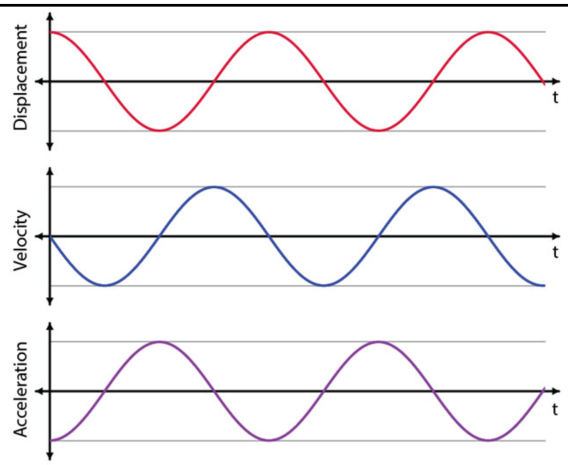
$$a = -a_{max}\cos(2\pi f t)$$

Where: $a_{max} = 4\pi^2 f^2 A$

- Since $x = A\cos(2\pi f t)$

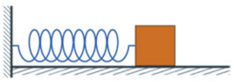
$$a = -4\pi^2 f^2 x$$

$$(a \propto -x)$$



Mass and Spring

- The restoring force for a mass on a spring is $F = -kx$.



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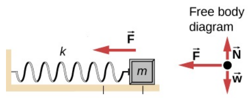

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Therefore $$ma = -kx$$

$$a = -\frac{k}{m}x$$

For simple harmonic motion $a = -4\pi^2 f^2 x$

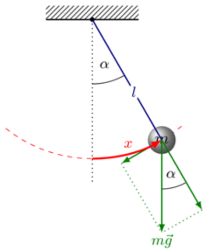
Therefore $$4\pi^2 f^2 = \frac{4\pi^2}{T^2} = \frac{k}{m}$$

Solving for T gives the period of a mass on a spring.

$$T_s = 2\pi\sqrt{\frac{m}{k}}$$

Simple Pendulum

- The restoring force is perpendicular to the pendulum bob and string



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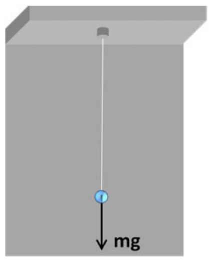

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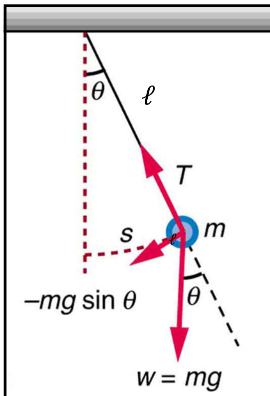


The restoring force is

$F = -mg \sin\theta$

For small angles $\sin\theta = \theta$

$F \approx -mg\theta$

Arc length s is

$s = \ell\theta$

This gives a force of

$$F = -\frac{mgs}{\ell}$$

This force is of the form $F \propto -x$ (s is the displacement x).

Therefore $F = ma = -\frac{mg}{\ell}x$

$$a = -\frac{g}{\ell}x$$

For simple harmonic motion $a = -4\pi^2 f^2 x$

Therefore $$4\pi^2 f^2 = \frac{4\pi^2}{T^2} = \frac{g}{\ell}$$

Solving for period T gives

$$T_p = 2\pi\sqrt{\frac{\ell}{g}}$$

as the period of a simple pendulum (with angle less than $\sim 15°$).

Energy

- In the case of undamped simple harmonic motion, the energy oscillates back and forth between kinetic and potential, going completely from one to the other as the system oscillates.
  - Energy in the system is conserved

$$K + U = constant$$

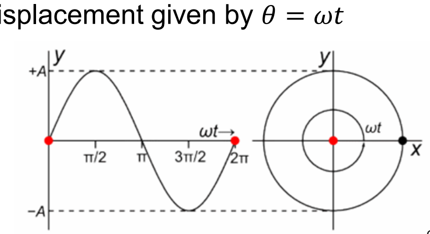
- The conservation of energy principle can be used to derive an expression for velocity at a given position.
  - At maximum displacement ($x = A$) all the energy is potential energy (maximum energy).

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$$

$$v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)}$$

SHM and Circular Motion

- SHM is related to circular motion
- An object moving in a circle with constant angular velocity has an angular displacement given by $\theta = \omega t$



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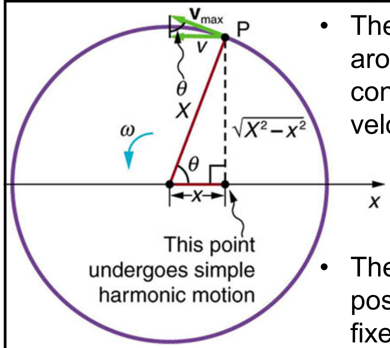

- The point P travels around the circle at constant angular velocity ω.
- The projection of the position of P onto a fixed axis undergoes simple harmonic motion.

- To see that the projection undergoes simple harmonic motion, note that its position x is given by $x = X\cos\theta$, where $\theta = \omega t$.
- This gives

$$x = X\cos\omega t$$

$$\omega = 2\pi f$$

$$x = X\cos(2\pi f t)$$

This is the equation for the position of a simple harmonic oscillator with an amplitude of X.